# PERSISTENCE OF WANDERING INTERVALS IN SELF-SIMILAR AFFINE INTERVAL EXCHANGE TRANSFORMATIONS

XAVIER BRESSAUD, PASCAL HUBERT, AND ALEJANDRO MAASS

Abstract. In this article we prove that given a self-similar interval exchange transformation $T_{(\lambda,\pi)}$, whose associated matrix verifies a quite general algebraic condition, there exists an affine interval exchange transformation with wandering intervals that is semi-conjugated to it. That is, in this context the existence of Denjoy counterexamples occurs very often, generalizing the result of M. Cobo in [C].

## 1. Introduction

Since the work of Denjoy [D] it is known that every $C^1$-diffeomorphism of the circle such that the logarithm of its derivative is a function of bounded variation has no wandering intervals. There is no analogous result for interval exchange transformations. Levitt in [L] found an example of a non-uniquely ergodic affine interval exchange transformation with wandering intervals. Latter, Camelier and Gutierrez [CG], using Rauzy induction technique exhibited a uniquely ergodic affine interval exchange transformation with wandering intervals. Moreover, this example is semi-conjugated to a self-similar interval exchange transformation. In geometric language, it means that this inter- val exchange transformation is induced by a pseudo-Anosov diffeomorphism. In combinatorial terms, the symbolic system is generated by a substitution
An interval exchange transformation (IET) is defined by the length of the intervals $\lambda = (\lambda_1, \dots, \lambda_r)$ and a permutation $\pi$. It is denoted by $T_{(\lambda,\pi)}$. To define an affine interval exchange transformation (AIET) one additional information is needed; the slope of the map on each interval. This is a vector $(w_1, \dots, w_r)$ with $w_i > 0$ for $i = 1, \dots, r$. Camelier and Gutierrez remarked that a necessary condition for an AIET to be conjugated to the interval exchange transformation $T_{(\lambda,\pi)}$ is that the vector $\log(w) = (\log(w_1), \dots, \log(w_r))$ is orthogonal to $\lambda$.
The conjugacy of an affine interval exchange transformation with an interval exchange transformation was studied in details by Cobo [C]. He proved that the regularity of the conjugacy depends on the position of the vector $\log(w)$ in the flag of the lyapunov exponents of the Rauzy-Veech-Zorich induction. In particular, assume that $T_{(\lambda,\pi)}$ is self-similar, which means that $\lambda$ is an eigenvector of a positive $r \times r$ matrix $R$ obtained by applying Rauzy induction a finite number of times. Cobo proves that if $\log(w)$ belongs to the contracting space of ${}^tR$ then $f$ is $C^1$ conjugated to $T_{(\lambda,\pi)}$. If $\log(w)$ is orthogonal to $\lambda$ and is not in the contracting

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space of ${}^tR$ then any conjugacy between $f$ and $T_{(\lambda,\pi)}$ is not an absolutely continuous function. Moreover, Camelier and Guttierez example shows that conjugacy between $f$ and $T_{(\lambda,\pi)}$ does not always exist.
In this paper, we prove the following result:

**Theorem 1.** *Let $T_{(\lambda,\pi)}$ be a self-similar interval exchange transformation and $R$ the associated matrix obtained by Rauzy induction. Let $\theta_1$ be the Perron-Frobenius eigenvalue of $R$. Assume that $R$ has an eigenvalue $\theta_2$ such that*

1. *$\theta_2$ is a conjugate of $\theta_1$,*
2. *$\theta_2$ is a real number,*
3. *$1 < \theta_2 (< \theta_1)$.*

*Then there exists an affine interval exchange transformation $f$ with wandering intervals that is semi-conjugated to $T_{(\lambda,\pi)}$.*

This result means that Denjoy counterexamples occur very often (see section 5).

1.1. **Reader's guide.** Camelier-Gutierrez [CG] and Cobo [C] developed an strategy to prove the existence of a wandering interval in an affine interval exchange transformation $f$ which is semi-conjugated with a given IET. We explain it in section 4. This strategy allowed them to achieve a first concrete example. Here we explore the limits of this method in order to consider a large (and in some sense abstract) family of IET. Let $T_{(\lambda,\pi)}$ be a self-similar interval exchange transformation with associated matrix $R$. Let $\gamma = (\gamma_1, \ldots, \gamma_r)$ be the vector of the logarithm of the slopes of the affine interval exchange transformation $f$. If $f$ admits a wandering interval $I$, the length $|f(I)|$ is equal to $e^{\gamma_j}|I|$ if $I$ is contained in interval $j$. Roughly speaking, to create a wandering interval from the interval exchange transformation $T_{(\lambda,\pi)}$, one blows up an orbit of $T_{(\lambda,\pi)}$. The difficulty is to insure that the total length remains finite. More precisely, if the symbolic coding of the orbit is $x = (x_n)_{n\in\mathbb{Z}}$, we have to check that the series

$$\sum_{n\geq 1} e^{-\gamma(x_0)-\ldots-\gamma(x_{n-1})} \text{ and } \sum_{n\geq 1} e^{\gamma(x_{-n})+\ldots+\gamma(x_{-1})} \tag{1.1}$$

converge. This is certainly not true for a generic point $x$ of the symbolic system associated to $T_{(\lambda,\pi)}$. Let $\ell(x)$ be the broken line with vertices $(n, \gamma(x_0) + \ldots + \gamma(x_{n-1}))_{n\in\mathbb{N}}$ and $(n, \gamma(x_{-n}) + \ldots + \gamma(x_{-1}))_{n\geq 1}$. Since $\gamma$ is orthogonal to $\lambda$, for a generic point $x$, the line $\ell(x)$ oscillates around 0 as predicted by Hálasz's Theorem ([Ha]). If the vector $\gamma$ is not in the contracting space of ${}^tR$ the amplitude of the oscillations tends to infinity with speed

$$n^{\log(\theta_1)/\log(\theta_2)}.$$

It is hoped that the series (1.1) converge if the $y$-coordinate of the broken line $\ell(x)$ is always positive and tends to infinity fast enough as $n$ tends to $\pm\infty$. Points with this property are called *minimal points*. Those are the main tool of the paper.
This analysis applies to a very large class of substitutions and not only to substitutions arising from interval exchange transformations. Section 3 gives an algorithm to construct minimal points. We prove that the prefix-suffix decomposition of any minimal point is ultimately periodic. From this analysis, we deduce that for any minimal point $x$ one has

$$\liminf_{n\to\infty} \frac{\gamma(x_0) + \ldots + \gamma(x_n)}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0 \text{ and } \liminf_{n\to\infty} \frac{-\gamma(x_{-n}) - \ldots - \gamma(x_{-1})}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0 \tag{1.2}$$

Formulas in (1.2) imply immediately the convergence of the series in (1.1). Moreover, formulas in (1.2) has its own interest. It is a strengthening of a result by Adamczewski [Ad] about discrepancy of substitutive systems.
Even if the fractal curves studied by Dumont and Thomas in [DT1], [DT2] are not considered explicitly in the article, they were a source of inspiration for the authors. These curves correspond to the renormalization of the broken lines $\ell(x)$ and appear in subsection 3.3 in another language.
In section 5, we discuss the hypothesis of the main result in a geometric language. We exhibit many examples where our hypothesis on the matrix $R$ are fulfilled.

## 2. Preliminaries

2.1. **Words and sequences.** Let $A$ be a finite set. One calls it an *alphabet* and its elements *symbols*. A *word* is a finite sequence of symbols in $A$, $w = w_0 \dots w_{\ell-1}$. The length of $w$ is denoted $|w| = \ell$. One also defines the empty word $\varepsilon$. The set of words in the alphabet $A$ is denoted $A^*$ and $A^+ = A^* \setminus \{\varepsilon\}$. We will need to consider words indexed by integer numbers, that is, $w = w_{-m} \dots w_{-1}.w_0 \dots w_\ell$ where $\ell, m \in \mathbb{N}$ and the dot separates negative and non-negative coordinates. If necessary we call them *dotted words*.
The set of one-sided infinite sequences $x = (x_i)_{i\in\mathbb{N}}$ in $A$ is denoted by $A^{\mathbb{N}}$. Analogously, $A^{\mathbb{Z}}$ is the set of two-sided infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$.
Given a sequence $x$ in $A^+$, $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ one denotes $x[i,j]$ the sub-word of $x$ appearing between indexes $i$ and $j$. Similarly one defines $x(-\infty, i]$ and $x[i,\infty)$. Let $w = w_{-m} \dots w_{-1}.w_0 \dots w_\ell$ be a (dotted) word in $A$. One defines the cylinder set $[w]$ as $\{x \in A^{\mathbb{Z}} : x[-m,\ell] = w\}$.
The shift map $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ or $T : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is given by $T(x) = (x_{i+1})_{i\in\mathbb{N}}$ for $x = (x_i)_{i\in\mathbb{N}}$. A subshift is any shift invariant and closed (for the product topology) subset of $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$. A subshift is minimal if all of its orbits by the shift are dense. In what follows we will use the shift map in several contexts, in particular restricted to a subshift. To simplify notations we keep the name $T$ all the time.

2.2. **Substitutions and minimal points.** We refer to [Qu] and [F] and references therein for the general theory of substitutions.
A *substitution* is a map $\sigma : A \to A^+$. It naturally extends to $A^+$, $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$; for $x = (x_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$ the extension is given by

$$\sigma(x) = \dots \sigma(x_{-2})\sigma(x_{-1}).\sigma(x_0)\sigma(x_1)\dots$$

where the central dot separates negative and non-negative coordinates of $x$. A further natural convention is that the image of the empty word $\varepsilon$ is $\varepsilon$.
Let $M$ be the matrix with indices in $A$ such that $M_{ab}$ is the number of times letter $b$ appears in $\sigma(a)$ for any $a, b \in A$. The substitution is primitive if there is $N > 0$ such that for any $a \in A$, $\sigma^N(a)$ contains any other letter of $A$ (here $\sigma^N$ means $N$ consecutive iterations of $\sigma$). Under primitivity one can assume without loss of generality that $M > 0$.
Let $X_\sigma \subseteq A^{\mathbb{Z}}$ be the subshift defined from $\sigma$. That is, $x \in X_\sigma$ if and only if any subword of $x$ is a subword of $\sigma^N(a)$ for some $N \in \mathbb{N}$ and $a \in A$.
Assume $\sigma$ is primitive. Given a point $x \in X_\sigma$ there exists a unique sequence $(p_i, c_i, s_i)_{i\in\mathbb{N}} \in (A^* \times A \times A^*)^{\mathbb{N}}$ such that for each $i \in \mathbb{N}$: $\sigma(c_{i+1}) = p_i c_i s_i$ and

$$\dots \sigma^3(p_3)\sigma^2(p_2)\sigma^1(p_1)p_0.c_0 s_0 \sigma^1(s_1)\sigma^2(s_2)\sigma^3(s_3)\dots$$

is the central part of $x$, where the dot separates negative and non-negative coordinates. This sequence is called the prefix-suffix decomposition of $x$ (see for instance [CS]).

If only finitely many suffixes $s_i$ are nonempty, then there exists $a \in A$ and non-negative integers $\ell$ and $q$ such that

$$x[0,\infty) = c_0 s_0 \sigma^1(s_1)\dots\sigma^\ell(s_\ell)\lim_{n\to\infty}\sigma^{nq}(a)$$

Analogously, if only finitely many $p_i$ are non empty, then

$$x(-\infty,-1] = \lim_{n\to\infty}\sigma^{np}(b)\sigma^m(p_m)\dots\sigma^1(p_1)p_0$$

for some $b \in A$ and non-negative integers $p$ and $m$.

Let $\theta_1$ be the Perron-Frobenius eigenvalue of $M$. Let $\lambda = (\lambda(a) : a \in A)^t$ be a strictly positive right eigenvector of $M$ associated to $\theta_1$. We will also assume the following algebraic property that we call (AH): $M$ has an eigenvalue $\theta_2$ which is a conjugate of $\theta_1$. Notice that this property coincides with hypothesis (1) of Theorem 1.

The following lemma are important consequences of the algebraic property (AH).

**Lemma 2.** *Let $\eta : \mathbb{Q}[\theta_1] \to \mathbb{Q}[\theta_2]$ be the field homomorphism that sends $\theta_1$ to $\theta_2$. The vector $\gamma = \eta(\lambda) = (\eta(\lambda(a)) : a \in A)^t$ is an eigenvector of $M$ associated to $\theta_2$.*

*Proof.* The field homomorphism $\eta$ naturally extends to $\mathbb{Q}[\theta_1]^{|A|}$. Since $\lambda$ belongs to $\mathbb{Q}[\theta_1]^{|A|}$ (up to normalization), then one deduces that $M\eta(\lambda) = \theta_2\eta(\lambda)$. Thus, $\eta(\lambda)$ is an eigenvector of $M$ associated to $\theta_2$. □

**Lemma 3.** *Let $\gamma$ be the eigenvector of $M$ associated to $\theta_2$ as in Lemma 2. Then for any $|A|$-tuple of non-negative integers $(n_a : a \in A)$, $\sum_{a\in A} n_a\gamma(a) = 0$ implies $n_a = 0$ for any $a \in A$.*

*Proof.* Assume $\sum_{a\in A} n_a\gamma(a) = 0$. Since $\gamma = \eta(\lambda)$, applying $\eta^{-1}$ one gets that $\sum_{a\in A} n_a\lambda(a) = 0$. This equality implies that $n_a = 0$ for every $a \in A$ because the coordinates of $\lambda$ are positive. □

Let $\gamma = \eta(\lambda)$ as in Lemma 2. For $w = w_0 \dots w_{l-1} \in A^+$ denote $\gamma(w) = \gamma(w_0) + \dots + \gamma(w_{l-1})$.

Let $x \in X_\sigma$. Define $\gamma_0(x) = 0$, $\gamma_n(x) = \sum_{i=0}^{n-1}\gamma(x_i)$ for $n > 0$ and $\gamma_n(x) = \sum_{i=n}^{-1}\gamma(x_i)$ for $n < 0$. Put $\Gamma(x) = \{\gamma_n(x) : n \in \mathbb{Z}\}$. In a similar way, given a (dotted) word $w = w_{-m}\dots w_0 \dots w_{l-1}$ one defines $\gamma_0(w) = 0$, $\gamma_n(w) = \sum_{i=0}^{n-1}\gamma(w_i)$ for $0 < n \le l$, $\gamma_n(w) = \sum_{i=n}^{-1}\gamma(w_i)$ for $-m \le n < 0$ and the set $\Gamma(w)$.

The best occurrence of a symbol $a \in A$ in $w$ is $-m \le i < l$ such that $w_i = a$ and $\gamma_{i+1}(w) = \min\{\gamma_{j+1}(w) : -m \le j < l, w_j = a\}$. By Lemma 3, under hypotheses (AH) this number is well defined and unique.

One says $x$ is *minimal* if $\gamma_n(x) \ge 0$ for any $n \in \mathbb{Z}$. The set of minimal points for $\sigma$ is denoted by $\mathcal{M}(\sigma)$. It is important to mention that if $x$ is a minimal point of a substitution satisfying hypothesis (AH) then, by Lemma 3, $\gamma_n(x) > 0$ whenever $n \neq 0$.

2.3. **Affine interval exchange transformations.** Let $0 = a_0 < a_1 < \dots < a_{r-1} < a_r = 1$ and $A = \{1,\dots,r\}$.

An *affine interval exchange transformation* (AIET) is a bijective map $f : [0,1) \to [0,1)$ of the form $f(t) = w_i t + v_i$ if $t \in [a_{i-1}, a_i)$ for $i \in A$. The vector $w =$

$(w_1, \ldots, w_r)$ is called the slope of $f$. We assume furthermore the slope is strictly positive.
An *interval exchange transformation* (IET) is an AIET with slope $w = (1, \ldots, 1)$. Commonly an IET is given by a vector $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_i = |a_i - a_{i-1}|$ for $i \in A$ and a permutation $\pi$ of $A$ which indicates the way intervals $[a_{i-1}, a_i)$'s are rearranged by the IET. Clearly, $a_i = \sum_{j=1}^{i} \lambda_j$. We use $T_{(\lambda,\pi)}$ to refer to the IET associated to $\lambda$ and $\pi$.
One says the AIET $f$ is semi-conjugated with the IET $T_{(\lambda,\pi)}$ if there is a monotonic, surjective and continuous map $h : [0,1) \to [0,1)$ such that $h \circ f = T_{(\lambda,\pi)} \circ h$.
Let $T_{(\lambda,\pi)}$ be an interval exchange transformation. There is a natural symbolic coding of the orbit of any point $t \in [0,1)$ by $T_{(\lambda,\pi)}$. Consider the partition $\alpha = \{[0, a_1), \ldots, [a_{i-1}, a_i), \ldots, [a_{r-1}, 1)\}$ and define $\phi(t) = (x_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$ by $t_i = j$ if and only if $T^i_{(\lambda,\pi)}(t) \in [a_{j-1}, a_j)$. The set $\phi([0,1))$ is invariant for the shift but it is not necessarily closed, then one considers its closure $X = \overline{\phi([0,1))}$. This procedure produces a semi-conjugacy (factor map) $\varphi : (X, T) \to ([0,1), T_{(\lambda,\pi)})$. If $t$ is not in the orbit of the extreme points $0, a_1, \ldots, 1$, then it has a unique preimage by $\varphi$. If not, it has at most two preimages corresponding to the coding of $(T^i_{(\lambda,\pi)}(\lim_{s\to t^-} s))_{i\in\mathbb{Z}}$.
We use freely concepts related to Rauzy-Zorich-Veech induction. Rauzy induction was defined in [Ra], extended to zippered rectangles by Veech [Ve], and accelerated by Zorich [Zo]. For a complete description about the Rauzy-Veech-Zorich induction see also the expository papers by Zorich [Zo2] and Yoccoz [Yo].
An IET $T_{(\lambda,\pi)}$ is *self-similar* if it can be recovered from itself after finitely many steps of Rauzy inductions (up to normalization). More precisely, there exists a loop in the Rauzy diagram and an associated *Perron-Frobenius* matrix $R$ such that

$$\theta_1 \lambda = R\lambda$$

with $\theta_1$ the dominant eigenvalue of $R$.
For a self-similar IET $T_{(\lambda,\pi)}$ there is a direct relation between the subshift $X$ and the matrix $R$ associated to $T_{(\lambda,\pi)}$. Indeed, there exists a substitution $\sigma : A \to A^+$ with associated matrix $M = {}^tR$ such that $X_\sigma = X$ (see [CG] and references therein). If the IET $T_{(\lambda,\pi)}$ is minimal then the subshift $X_\sigma$ is minimal too. In the sequel, we will use the fact that the substitution $\sigma$ is primitive which implies that $X_\sigma$ is minimal. Nevertheless, no specific property of substitutions obtained from $T_{(\lambda,\pi)}$ will be needed for our purpose.
The relation between self-similar IET and pseudo-Anosov diffeomorphisms is explained in [Ve].

## 3. Construction of minimal points

Let $\sigma : A \to A^+$ be a primitive substitution with associated matrix $M > 0$. Let $\theta_1$, $\theta_2$, $\lambda$ and $\gamma$ be as in subsection 2.2. In addition, assume $\theta_2$ verifies the hypotheses of Theorem 1. By Perron-Frobenius theorem, $\gamma$ has negative and positive coordinates. The main objective of the section is to give a combinatorial construction of minimal points in this case.

3.1. **Existence of minimal points.**

**Lemma 4.** *Let $a \in A$ such that $\gamma(a) > 0$ and $n \in \mathbb{N}$. Write $\sigma^n(a) = p_n s_n$ where the minimum of $\Gamma(\sigma^n(a))$ is attained at $\gamma_i(\sigma^n(a))$ and $i = |p_n|$. Then $\gamma(s_n) \geq \theta_2^n \gamma(a)$. In particular $|s_n|$ grows exponentially fast with $n$.*

*Proof.* Observe that $\gamma(p_n) + \gamma(s_n) = \theta_2^n \gamma(a)$ and $\gamma(p_n) \leq 0$. □

**Lemma 5.** $\mathcal{M}(\sigma) \neq \emptyset$

*Proof.* Since $\gamma$ has positive and negative coordinates and $X_\sigma$ is minimal, then there exist $b, c \in A$ such that $bc$ is a subword of a point in $X_\sigma$ and $\gamma(b) < 0, \gamma(c) > 0$ holds.
Let $n \geq 0$ and define $u_n = \sigma^n(b).\sigma^n(c)$. The sequence $\Gamma(u_n)$ attains its minimum at some $N_n \in \{-|\sigma^n(b)|, \ldots, -1, 0, \ldots, |\sigma^n(c)|\}$. Define the (dotted)word $v_n = u_n[-|\sigma^n(b)|, N_n - 1].u_n[N_n, |\sigma^n(c)| - 1] = v_n^-.v_n^+$. The minimum of $\Gamma(v_n)$ is attained at coordinate 0, and is equal to 0.
By Lemma 4 there is a subsequence $(n_i)_{i \in \mathbb{N}}$ such that

$$\lim_{i \to \infty} |v_{n_i}^-| = \lim_{i \to \infty} |v_{n_i}^+| = \infty$$

By compactness and eventually taking once again a subsequence there exists $x \in X$ such that for any $m \in \mathbb{N}$ there is $i \in \mathbb{N}$ with $n_i \geq m$ and $x \in [v_{n_i}^-.v_{n_i}^+]$. Thus $\Gamma(x[-m, m]) \subseteq \mathbb{R}^+$ and its minimum is zero at zero coordinate. This implies $x \in \mathcal{M}(\sigma)$. □

3.2. **The** ***best strategy*** **algorithm.** In what follows we develop a procedure to construct *minimal* points that will become useful in next subsections.
The following two lemma follow directly from equality $M\gamma = \theta_2 \gamma$. Their simple proofs are left to the reader.

**Lemma 6.** *Let $m \in \mathbb{N}$ and $w \in A^+$. Then $\gamma(\sigma^m(w)) = \theta_2^m \gamma(w)$.*

**Lemma 7.** *Let $w = w_0 \ldots w_{l-1} \in A^+$. Write $\sigma(w) = \sigma(w_0) \ldots \sigma(w_{l-1})$. The minimum of $\Gamma(\sigma(w))$ is attained in a coordinate corresponding to some $\sigma(w_i)$, where $w_i$ is the best occurrence of this symbol in $w$.*

3.2.1. *The basic procedure.* The following procedure will allow to construct the prefix-suffix decomposition of a minimal point.
**Step 0:** For each $a \in A$ write $\sigma(a) = p_0^{a,0} c_0^{a,0} s_0^{a,0}$ where $\Gamma(\sigma(a))$ attains its minimum at $\gamma_{|p_0(a)|}(\sigma(a))$.
**Step 1:** Let $a \in A$. By Lemma 7, the minimum of $\Gamma(\sigma^2(a))$ comes from $\sigma(b)$ for some $b \in A$ in its best occurrence in $\sigma(a)$. Write $\sigma(a) = p_1^{a,1} c_1^{a,1} s_1^{a,1}$ where $c_1^{a,1} = b$ is the best occurrence of $b$ in $\sigma(a)$. Put $w_1(a) = \sigma(p_1^{a,1}) p_0^{b,0}.c_0^{b,0} s_0^{b,0} \sigma(s_1^{a,1})$, where the dot separates negative and non-negative coordinates. Let $p_0^{a,1} = p_0^{b,0}$, $c_0^{a,1} = c_0^{b,0}$ and $s_0^{a,1} = s_0^{b,0}$. The sequence $(p_i^{a,1}, c_i^{a,1}, s_i^{a,1})_{i=0}^1$ is called the best strategy for symbol $a$ at step 1. By construction $\Gamma(w_1(a)) \subseteq \mathbb{R}^+$ and the minimum is equal to zero at coordinate zero.
**Step n+1:** assume in previous step we have constructed for each symbol $a \in A$ the best strategy $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$. This sequence verifies:
(i) for $0 \leq i \leq n$, $\sigma(c_{i+1}^{a,n}) = p_i^{a,n} c_i^{a,n} s_i^{a,n}$ (here $c_{n+1}^{a,n} = a$). Moreover, each $c_i^{a,n}$ is the best occurrence of this symbol in $\sigma(c_{i+1}^{a,n})$.

(ii) $\Gamma(w_n(a)) \subseteq \mathbb{R}^+$ and its minimum is zero at zero coordinate, where

$$w_n(a) = \sigma^{n+1}(a) = \sigma^n(p_n^{a,n}) \dots \sigma(p_1^{a,n}) p_0^{a,n}.c_0^{a,n} s_0^{a,n} \sigma(s_1^{a,n}) \dots \sigma^n(s_n^{a,n})$$

Now we proceed as in step 1. Consider $a \in A$. By Lemma 7, the minimum of $\Gamma(\sigma^{n+2}(a))$ comes from $\sigma^{n+1}(b)$ for some $b \in A$ in its best occurrence in $\sigma(a)$. Write $\sigma(a) = p_{n+1}^{a,n+1} c_{n+1}^{a,n+1} s_{n+1}^{a,n+1}$ where $c_{n+1}^{a,n+1} = b$ is the best occurrence of $b$ in $\sigma(a)$. The finite sequence $(p_i^{a,n+1}, c_i^{a,n+1}, s_i^{a,n+1})_{i=0}^{n+1}$ where $(p_i^{a,n+1}, c_i^{a,n+1}, s_i^{a,n+1}) = (p_i^{b,n}, c_i^{b,n}, s_i^{b,n})$ for $0 \leq i \leq n$ is a best strategy for $a$ at step $n+1$ and verifies conditions (i) and (ii) by construction.

3.2.2. *Finitely many minimal points.* For each $a \in A$ and $n \in \mathbb{N}$ consider the cylinder set $C^{a,n} = [w_n(a)]$, where $w_n(a)$ is the dotted word defined in previous subsection. It is clear from the basic procedure that for any $a \in A$ and $n \in \mathbb{N}$ there exists a unique $b \in A$ such that $C^{a,n+1} \subseteq C^{b,n}$. Thus, by compactness, there exist at most $|A|$ infinite decreasing sequences of the form $(C^{a_n,n})_{n\in\mathbb{N}}$. Let $C_1, \dots, C_\ell$ with $\ell \leq |A|$ be the collection of intersections of such sequences. Remark that such sets are finite.
Given a minimal point $x \in X$ with prefix-suffix decomposition $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ and $n \in \mathbb{N}$, there is $a_n \in A$ such that $(p_i, c_i, s_i) = (p_i^{a_n,n}, c_i^{a_n,n}, s_i^{a_n,n})$ for $0 \leq i \leq n$. Therefore, $x \in C_i = \bigcap_{n\in\mathbb{N}} C^{a_n,n}$ for some $1 \leq i \leq \ell$.
The following proposition is plain.

**Proposition 8.** *There are finitely many minimal points.*

We will see later that minimal points have ultimately periodic prefix-suffixe decomposition. This fact yields to an alternative proof of previous proposition.

3.3. **Serie associated to a minimal point.** Define $\overline{\mathcal{S}} = \{(p_i, c_i, s_i)_{i\in\mathbb{N}} : \forall\ i > 0,\ \sigma(c_i) = p_{i-1}c_{i-1}s_{i-1}\}$ and $\underline{\mathcal{S}} = \{(p_i, c_i, s_i)_{i\in\mathbb{N}} : \forall\ i \geq 0,\ \sigma(c_i) = p_{i+1}c_{i+1}s_{i+1}\}$. Observe that finite sequences taken from sequences in $\overline{\mathcal{S}}$ and $\underline{\mathcal{S}}$ coincide once reversed.
Let $a \in A$ and $n \geq 1$. Then $\sigma^n(a)$ can be decomposed as

$$\sigma^n(a) = \sigma^{n-1}(p_1) \dots \sigma(p_{n-1}) p_n c_n s_n \sigma(s_{n-1}) \dots \sigma^{n-1}(s_1)$$

where for all $1 \leq i \leq n$, $\sigma(c_{i-1}) = p_i c_i s_i$ (we have considered $c_0 = a$). This decomposition is not unique. To $a$ and the finite sequence $(p_i, c_i, s_i)_{i=1}^n$ one associates the finite sum:

$$v(a; (p_i, c_i, s_i)_{i=1}^n) = \sum_{i=1}^n \theta_2^{-i} \gamma(p_i)$$

Clearly, given $\mathbf{x} = (p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{x}} = a$, the series

$$v(a; \mathbf{x}) = \lim_{n\to\infty} v(a; (p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}})_{i=1}^n) = \sum_{i\geq 1} \theta_2^{-i} \gamma(p_i^{\mathbf{x}})$$

exists.
Let $v(a) = \min\{v(a; \mathbf{x}) : \mathbf{x} \in \underline{\mathcal{S}} \text{ with } c_0^{\mathbf{x}} = a\}$. A sequence $\mathbf{x} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{x}} = a$ such that $v(a; \mathbf{x}) = v(a)$ is said to be *minimal for a*.
The best strategy for symbol $a$ at step $n \geq 1$ given by the algorithm produces a finite sequence $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$. Set $v_n(a) = \sum_{i=0}^n \theta^{-n+i-1} \gamma(p_i^{a,n})$. It follows that $v_n(a) = v(a; (p_{n-i}^{a,n}, c_{n-i}^{a,n}, s_{n-i}^{a,n})_{i=0}^n)$.

**Lemma 9.** *For every $a \in A$ and $n \geq 1$, $v_n(a)$ is minimal among the $v(a;(p_i,c_i,s_i)_{i=1}^{n+1})$ and $v(a) = \lim_{n\to\infty} v_n(a)$.*

*Proof.* The first fact is analogous to say that $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$ is the best strategy. Moreover, $|v_n(a)-v(a)| \leq K\theta_2^{-n}$ for some constant $K>0$. This implies the desired result. $\square$

**Lemma 10.** *Let $a \in A$. Assume there is a finite sequence $(\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$ such that for infinitely many $n \in \mathbb{N}$, $(p_{n-j+1}^{a,n}, c_{n-j+1}^{a,n}, s_{n-j+1}^{a,n})_{j=1}^l = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$. Then, there exists $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ such that $(\mathbf{y}_j)_{j=1}^l = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$, $c_0^{\mathbf{y}} = a$ and $v(a) = v(c_0^{\mathbf{y}};\mathbf{y})$.*

*Proof.* For any $n \in \mathbb{N}$ where the property of the lemma holds consider the point

$$\mathbf{y}^{(n)} = \mathbf{y}_0^{(n)} \ldots \mathbf{y}_n^{(n)} = (p,a,s)(p_n^{a,n}, c_n^{a,n}, s_n^{a,n}) \ldots (p_0^{a,n}, c_0^{a,n}, s_0^{a,n})$$

where $\sigma(b) = pas$ for some $b \in A$.
Let $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}}$ be the limit of a subsequence $(\mathbf{y}^{(n_i)})_{i\in\mathbb{N}}$. It follows by construction that $(\mathbf{y}_j)_{j=1}^l = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$, $c_0^{\mathbf{y}} = a$ and $\sigma(c_i^{\mathbf{y}}) = p_{i+1}^{\mathbf{y}} c_{i+1}^{\mathbf{y}} s_{i+1}^{\mathbf{y}}$ for any $i \geq 0$. Also, $c_{i+1}^{\mathbf{y}}$ is the best occurrence of this symbol in $\sigma(c_i^{\mathbf{y}})$.
Let $\epsilon > 0$ and $i_0 \in \mathbb{N}$ such that $|v(a) - v_{n_i}(a)| \leq \epsilon/2$ for $i \geq i_0$. Let $L \in \mathbb{N}$ be such that $\theta_2^{-L} \leq \epsilon/4C$ where $C > 0$ is such that $|\gamma(p_i^{c,n})|/(\theta_2 - 1) \leq C$ for any $c \in A$ and $n \in \mathbb{N}$. Thus for $i$ enough large, $(p_j^{\mathbf{y}}, c_j^{\mathbf{y}}, s_j^{\mathbf{y}}) = (p_{n_i-j+1}, c_{n_i-j+1}, s_{n_i-j+1})$ for $0 \leq j < L$ and $|v(a) - \sum_{i\geq 1} \theta_2^{-i}\gamma(p_i^{\mathbf{y}})| \leq \epsilon$. Since $\epsilon$ is arbitrary one concludes $v(c_0^{\mathbf{y}}) = v(a) = \sum_{i\geq 1}\theta_2^{-i}\gamma(p_i^{\mathbf{y}})$. $\square$

One says that a point $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ verifies the *continuation property* if $v(c_i^{\mathbf{y}}) = v(c_i^{\mathbf{y}}; T^i(\mathbf{y}))$ for all $i \geq 0$, where $T$ is the shift map. It is clear that $T^i(\mathbf{y})$ has the continuation property too, for any $i \in \mathbb{N}$. In fact to satisfy the continuation property it is enough to be minimal for $c_0^{\mathbf{y}}$.

**Lemma 11.** *If $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ is minimal for $c_0^{\mathbf{y}}$ (that is, $v(c_0^{\mathbf{y}}) = v(c_0^{\mathbf{y}};\mathbf{y})$) then $\mathbf{y}$ verifies the continuation property.*

*Proof.* Let $b = c_1^{\mathbf{y}}$ and $\mathbf{z} = (p_i^{\mathbf{z}}, c_i^{\mathbf{z}}, s_i^{\mathbf{z}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{z}} = b$ and $v(b;\mathbf{z}) = v(b)$ given by Lemma 10 (considering $l = 0$). The sequence $\mathbf{w} = \mathbf{y}_0\mathbf{y}_1 T(\mathbf{z})$ belongs to $\underline{\mathcal{S}}$ and verifies $v(a;\mathbf{w}) = \theta_2^{-1}\gamma(p_1^{\mathbf{y}}) + \theta_2^{-1}v(b)$. Thus, if $v(b;T(\mathbf{y})) > v(b)$, from $v(a) = v(a;\mathbf{y}) = \theta_2^{-1}\gamma(p_1^{\mathbf{y}}) + \theta_2^{-1}v(b;T(\mathbf{y}))$, one deduces that $v(a;\mathbf{w}) < v(a)$ which is a contradiction. $\square$

This lemma proves that sequences $\mathbf{y}$ constructed in Lemma 10 verifies the continuation property.

3.4. **Minimal points are ultimately periodic.** In this section we prove that any minimal point $x \in X_\sigma$ has ultimately periodic prefix-suffix decomposition. That is, if $\bar{x} = (p_i, c_i, s_i)_{i\in\mathbb{N}}$ is the prefix-suffix decomposition of $x$, then $T^{p+q}(\bar{x}) = T^q\bar{x}$ for some $p > q \geq 0$. If $q = 0$ one says $x$ is a periodic minimal point.

**Lemma 12.** *For every $a \in A$ there exists a ultimately periodic point $\mathbf{x}(a) = (p_i^{\mathbf{x}(a)}, c_i^{\mathbf{x}(a)}, s_i^{\mathbf{x}(a)})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{x}(a)} = a$ and $v(a;\mathbf{x}(a)) = v(a)$ (so, $\mathbf{x}(a)$ has the continuation property).*

*Proof.* Let $a \in A$ and $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{S}$ with $c_0^{\mathbf{y}} = a$ and $v(a;\mathbf{y}) = v(a)$ given by Lemma 10 (considering $l = 0$). We are going to construct another one with ultimately periodic decomposition.
Let $0 < q < p$ be such that $\mathbf{y}_q = \mathbf{y}_p$ and $c_{q-1}^{\mathbf{y}} = c_{p-1}^{\mathbf{y}} = b$. The preperiodic sequence $\mathbf{x} = \mathbf{y}_0 \dots \mathbf{y}_{q-1}\mathbf{y}_q \dots \mathbf{y}_{p-1}\mathbf{y}_q \dots \mathbf{y}_{p-1} \dots \in \underline{S}$ since $\sigma(c_{p-1}^{\mathbf{y}}) = p_q^{\mathbf{y}} c_q^{\mathbf{y}} s_q^{\mathbf{y}}$ by hypothesis. We are going to prove that $v(a;\mathbf{x}) = v(a)$.
Observe that, by Lemma 10,

$$v(b) = \sum_{i\geq q} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) \text{ and } v(b) = \sum_{i\geq p} \theta_2^{-(i-p+1)}\gamma(p_i^{\mathbf{y}}) \ .$$

Thus, $v(b) = \sum_{i=q}^{p-1} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) + \sum_{i\geq p} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) = \sum_{i=q}^{p-1} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) + \theta_2^{-(p-q)}v(b)$. If we denote $B = \sum_{i=q}^{p-1} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}})$, then $v(b) = B\sum_{i\geq 0}\theta_2^{-(p-q)i}$. Consequently,

$$v(a) = \sum_{i=1}^{q-1} \theta_2^{-i}\gamma(p_i^{\mathbf{y}}) + \theta_2^{-(q-1)}B\sum_{i\geq 0}\theta_2^{-(p-q)i}$$

On the other hand, a direct computation yields to

$$v(a;\mathbf{x}) = \sum_{i=1}^{q-1} \theta_2^{-i}\gamma(p_i^{\mathbf{y}}) + \theta_2^{-(q-1)}(\sum_{i\geq 0}\theta_2^{-(p-q)i}B) \ ,$$

which implies, $v(a;\mathbf{x}) = v(a)$. □

To each preperiodic sequence $\mathbf{x}(a)$ constructed in previous lemma one can associate a point $x$ in the symbolic space $X_\sigma$ with periodic prefix-suffix decomposition of period

$$(p_0, c_0, s_0), \dots, (p_{p-q}, c_{p-q}, s_{p-q}) = (p_{p-1}^{\mathbf{x}(a)}, c_{p-1}^{\mathbf{x}(a)}, s_{p-1}^{\mathbf{x}(a)}), \dots, (p_q^{\mathbf{x}(a)}, c_q^{\mathbf{x}(a)}, s_q^{\mathbf{x}(a)}) \ .$$

Even if, by construction, this point is associated to the minimal value $v(b)$, there is no reason for it to be a minimal point.
Without loss of generality we will do the following simplification. By iterating $\sigma$ enough times one can assume that all ultimately periodic sequences constructed in Lemma 12 are of period 1 and of preperiod 1. That is, for each letter $a \in A$, $c_0^{\mathbf{x}(a)} = a$ and $\mathbf{x}_i = (p^{(a)}, \hat{a}, s^{(a)})$ for all $i \geq 1$. The letter $a \in A$ is *periodic* if $\hat{a} = a$ and one denotes $\hat{A}$ the subset of periodic letters. Since, the construction of Lemma 12 implies that $v(c_i^{\mathbf{x}(a)}) = v(a; T^i(\mathbf{x}(a)))$ for $0 \leq i \leq p-1$, then under this simplification $v(\hat{a}) = v(\hat{a}; T(\mathbf{x}(a)))$.

**Lemma 13.** *Let $\mathbf{y} \in \underline{S}$ verifying the continuation property. Then, for any $i \geq 1$ the point $\mathbf{y}^{(i)} = \mathbf{y}_0 \dots \mathbf{y}_i T(\mathbf{x}(c_i^{\mathbf{y}}))$ has the continuation property too.*

*Proof.* Let $i \geq 1$ and $1 \leq j \leq i$. From the continuation property one deduces that $v(c_j^{\mathbf{y}}) = \sum_{k=1}^{i-j} \theta_2^{-k}\gamma(p_{k+j}^{\mathbf{y}}) + \theta_2^{-(i-j)}v(c_i^{\mathbf{y}})$. But, $v(c_i^{\mathbf{y}}) = v(c_i^{\mathbf{y}}; \mathbf{x}(c_i^{\mathbf{y}}))$ and $v(\hat{c}_i^{\mathbf{y}}) = v(\hat{c}_i^{\mathbf{y}}; T(\mathbf{x}(c_i^{\mathbf{y}})))$, then $\mathbf{y}^{(i)} = \mathbf{y}_0 \dots \mathbf{y}_{i-1} T(\mathbf{x}(c_i^{\mathbf{y}}))$ has the continuation property too. □

**Lemma 14.** *Let $\mathbf{x}, \mathbf{y} \in \underline{S}$ such that $(\mathbf{x}_i)_{i\geq l+1} = (\mathbf{y}_i)_{i\geq l+1}$ and $c_0^{\mathbf{x}} = c_0^{\mathbf{y}} = a$. If $v(a;\mathbf{x}) = v(a;\mathbf{y})$ then $(\mathbf{x}_i)_{i\geq 1} = (\mathbf{y}_i)_{i\geq 1}$.*

*Proof.* Let $\mathbf{x} = (p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}})_{i\in\mathbb{N}}$ and $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}}$. From the hypothesis one deduces that

$$\sum_{i=1}^{l} \theta_2^{-i}\gamma(p_i^{\mathbf{x}}) = \sum_{i=1}^{l} \theta_2^{-i}\gamma(p_i^{\mathbf{y}})$$

and consequently

$$\gamma(\sigma^{l-1}(p_1^{\mathbf{x}})\dots p_l^{\mathbf{x}}) = \gamma(\sigma^{l-1}(p_1^{\mathbf{y}})\dots p_l^{\mathbf{y}}) \ .$$

But words $\sigma^{l-1}(p_1^{\mathbf{x}})\dots p_l^{\mathbf{x}}$ and $\sigma^{l-1}(p_1^{\mathbf{y}})\dots p_l^{\mathbf{y}}$ are prefixes of $\sigma^l(a)$. Then, by the algebraic condition (Lemma 3) they must be the same. This implies $(p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}}) = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})$ for $1 \le i \le l$. □

**Theorem 15.** *The prefix-suffix decomposition of any minimal point is ultimately periodic.*

*Proof.* Let $x \in X_\sigma$ be minimal point with prefix-suffixe decomposition $(p_i, c_i, s_i)_{i\in\mathbb{N}}$. There exists a finite sequence $(\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=0}^{l}$ such that $(\bar{p}_0, \bar{c}_0, \bar{s}_0) = (\bar{p}_l, \bar{c}_l, \bar{s}_l)$ and for infinitely many $i \in \mathbb{N}$, $(p_{i-j}, c_{i-j}, s_{i-j})_{j=0}^{l} = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=0}^{l}$.
Let $a = \bar{c}_0 = \bar{c}_l$. By Lemma 10, there is a point $\mathbf{y} \in \underline{\mathcal{S}}$ verifying the continuation property such that $(\mathbf{y}_j)_{j=0}^{l} = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=0}^{l}$. In particular, $v(a;\mathbf{y}) = v(a)$ and $v(a;T^l(\mathbf{y})) = v(a)$. Since, $v(a) = v(a;\mathbf{x}(a))$, then by Lemma 13 the sequence $\mathbf{z} = \mathbf{y}_0 \dots \mathbf{y}_l T(\mathbf{x}(a))$ has the continuation property and $v(a) = v(a;\mathbf{z})$ holds. Therefore, by Lemma 14, one concludes that $(\mathbf{x}(a))_{i\ge1} = (\mathbf{z}_i)_{i\ge1}$.
We have proved that $a \in \hat{A}$, that is $a = \hat{a}$, and that the word $(p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})$ appears infinitely many times in the prefix-suffixe decomposition of $x$. Now we prove that $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ is ultimately periodic with period $(p^{(a)}, a, s^{(a)})$.
Assume this result does not hold. Then there is $b \neq a$ in $A$ such that

$$(p_i, c_i, s_i)(p_{i-1}, c_{i-1}, s_{i-1})(p_{i-2}, c_{i-2}, s_{i-2}) = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})(p, b, s)$$

for infinitely many $i \in \mathbb{N}$.
By Lemma 10, there is a point $\mathbf{w} \in \underline{\mathcal{S}}$ verifying the continuation property and such that $\mathbf{w}_0\mathbf{w}_1\mathbf{w}_2 = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})(p, b, s)$. Since $v(b) = v(b;T^2(\mathbf{w}))$ and $v(b) = v(b;\mathbf{x}(b))$, by Lemma 13, the points $\mathbf{u} = \mathbf{w}_0\mathbf{w}_1\mathbf{w}_2 T(\mathbf{x}(b))$ and $\mathbf{v} = \mathbf{x}(a)_0\mathbf{x}(b)$ have the continuation property. Since $\mathbf{u}$ and $\mathbf{v}$ are ultimately equal, then, by Lemma 14, one concludes $a = b$ which is a contradiction. This proves the theorem. □

We stress the fact that it is possible to construct examples with minimal points having ultimately periodic but not periodic prefix-suffix decomposition.

### 3.5. **Convergence of series associated to minimal points.**

**Lemma 16.** *Let $\mathbf{y} \in \underline{\mathcal{S}}$ such that $c_0^{\mathbf{y}} = a \in \hat{A}$ and $v(a;\mathbf{y}) = v(a)$. Then, $\mathbf{y}_1 = (p^{(a)}, a, s^{(a)})$.*

*Proof.* Put $c_0^{\mathbf{y}} = a$. First we prove that $v(c_1^{\mathbf{y}}) = v(c_1^{\mathbf{y}};T(\mathbf{y}))$. Let $\mathbf{z} = \mathbf{y}_0\mathbf{y}_1 T(\mathbf{x}(c_1^{\mathbf{y}})) \in \underline{\mathcal{S}}$. If the assertion is not true then

$$v(a) = \theta_2^{-1}(\gamma(p_1^{\mathbf{y}}) + v(c_1^{\mathbf{y}};T(\mathbf{y}))) > \theta_2^{-1}(\gamma(p_1^{\mathbf{y}}) + v(c_1^{\mathbf{y}})) = v(a;\mathbf{z}) \ge v(a)$$

which is a contradiction. Thus, $v(c_1^{\mathbf{y}}) = v(c_1^{\mathbf{y}};T(\mathbf{y}))$ and furthermore $v(a) = v(a;\mathbf{z})$. Then, the point $\mathbf{w} = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})T(\mathbf{z})$ verifies $v(a) = v(a;\mathbf{w})$. But $\mathbf{w}$ and $\mathbf{x}(a)$ are ultimately equal, then by Lemma 14, $\mathbf{y}_1 = (p^{(a)}, a, s^{(a)})$.

□

**Lemma 17.** *Let $x \in X_\sigma$ be a minimal point. Then,*

$$\liminf_{n\to\infty} \frac{\gamma(x_0 \dots x_n)}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0 \text{ and } \liminf_{n\to\infty} \frac{-\gamma(x_{-n} \dots x_{-1})}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0$$

*Proof.* We only prove the first inequality, the other one can be shown analogously. Assume the result does not hold. Then, for a subsequence $(n_i)_{i\in\mathbb{N}}$,

$$\lim_{i\to\infty} \frac{\gamma(x_0 \dots x_{n_i})}{n_i^{\frac{\log(\theta_2)}{\log(\theta_1)}}} = 0$$

Let $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ be the prefix-suffix decomposition of $x$ and let $a \in \hat{A}$ such that $(p^{(a)}, a, s^{(a)})$ is the periodic part of it.
(1) First we assume $s^{(a)}$ is different from the empty word. Let $N_i$ be the minimal integer such that $x_1 \dots x_{n_i}$ is the prefix of $\sigma^{N_i}(a)$.
Consider the prefix-suffix decomposition $(p_j^{(n_i)}, c_j^{(n_i)}, s_j^{(n_i)})_{j\in\mathbb{N}}$ of $T^{n_i+1}(x)$. Clearly,

$$\sigma^{N_i-1}(p_{N_i-1}^{(n_i)}) \dots \sigma(p_1^{(n_i)})p_0^{(n_i)} = \sigma^{N_i-1}(p_{N_i-1}) \dots \sigma(p_1)p_0 x_0 \dots x_{n_i}$$

Then,

$$\sum_{j=N_i-1}^{0} \theta_2^j \gamma(p_j^{(n_i)}) = \sum_{j=N_i-1}^{0} \theta_2^j \gamma(p_j) + \gamma(x_0 \dots x_{n_i})$$

Dividing by $\theta_2^{N_i}$ one gets,

$$\sum_{j=1}^{N_i} \theta_2^{-j} \gamma(p_{N_i-j}^{(n_i)}) = \sum_{j=1}^{N_i} \theta_2^{-j} \gamma(p_{N_i-j}) + \theta_2^{-N_i} \gamma(x_0 \dots x_{n_i})$$

Taking the limit when $i \to \infty$ and using the fact that $x$ is minimal one gets

$$\lim_{i\to\infty} \sum_{j=1}^{N_i} \theta_2^{-j} \gamma(p_{N_i-j}^{(n_i)}) = v(a)$$

since by assumption $\lim_{i\to\infty} \theta_2^{-N_i} \gamma(x_0 \dots x_{n_i}) = 0$. Observe that $n_i$ behaves like $\theta_1^{N_i}$.
This property allows to show, following the same ideas used to prove Lemma 10, that there is $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ such that $v(a; \mathbf{y}) = v(a)$. By Lemma 16, $\mathbf{y}_1 = (p^{(a)}, a, s^{(a)})$. This implies $n_i + 1 = 0$ for some large $i$, which is a contradiction.
(2) Now suppose $s^{(a)}$ is the empty word. Then, (considering a power of $\sigma$ if necessary) $(x_n)_{n\geq N} = \lim_{m\to\infty} \sigma^m(b)$ for some $N \in \mathbb{N}$ and $b \in A$. If we write $\sigma(b) = bs$ one obtains $x_m \dots = bs\sigma(s)\sigma^2(s) \dots$.
We claim $v(b) = 0$. Suppose this is not true. Then for $k \in \mathbb{N}$ large enough one has $\sum_{i=1}^{k} \theta_2^{K-i} \gamma(p_i^{\mathbf{x}(b)}) \leq K\theta_2^k$ with $K < 0$. That is, $\gamma$ applied to a prefix of $\sigma^k(b)$ can be as negative as we want if $k$ increases. This implies that $\gamma_n(x) < 0$ for some $n \in \mathbb{N}$, which is imposible since $x$ is a minimal point. Then $v(b) = 0$. Furthermore, we have proved that $\gamma(x_N \dots x_{N+i}) > 0$ for all $i \geq 1$. One also deduces, by the algebraic condition, that $\mathbf{x}(b) = (\varepsilon, b, s)_{i\in\mathbb{N}}$, where $\varepsilon$ is the empty word.
To conclude one uses part (1) with $b$ instead of $a$.

∎

The following proposition is plain.

**Proposition 18.** *Let $x \in X_\sigma$ be a minimal point. Then,*

$$\sum_{n\geq 1} e^{-\gamma(x_0\dots x_{n-1})} < \infty \text{ and } \sum_{n\geq 1} e^{\gamma(x_{-n}\dots x_{-1})} < \infty$$

## 4. Proof of the Main Theorem

The arguments of this section follows the strategy developed in the works of [CG] and [C].

Let $T_{(\lambda,\pi)}$ be a self-similar interval exchange transformation and $R$ its associated matrix. Assume $R$ verifies hypotheses of Theorem 1.

Let $X_\sigma$ be the substitutive system associated to $T_{(\lambda,\pi)}$ and let $M = {}^tR$ be the associated matrix. Consider a minimal point $x \in X_\sigma$. By Proposition 18,

$$K = \sum_{n\geq 1} e^{\gamma(x_{-n}\dots x_{-1})} + 1 + \sum_{n\geq 1} e^{-\gamma(x_0\dots x_{n-1})} < \infty$$

Let $t = \varphi(x)$. That is, $x$ is the coding of $t$ or $x$ is the coding of $(\lim_{s\to t^-} T^i(s))_{i\in\mathbb{Z}}$ in the case $t$ is in the orbit of one of the $a_i$'s. To simplify notations we assume the first case holds, the other one is analogous.

Define the probability measure $\mu_t$ on $[0,1)$ by

$$\mu_t = \frac{1}{K}\left(\sum_{n\geq 1} e^{\gamma(x_{-n}\dots x_{-1})}\delta_{T^{-n}_{(\lambda,\pi)}t} + \delta_t + \sum_{n\geq 1} e^{-\gamma(x_0\dots x_{n-1})}\delta_{T^{n}_{(\lambda,\pi)}t}\right)$$

**Lemma 19.** *For every Borel set $I \subseteq [0,1)$*

$$\mu_t(T_{(\lambda,\pi)}(I)) = \sum_{i=1}^{r} e^{-\gamma_i}\mu_t(I\cap[a_{i-1},a_i))$$

*Proof.* It is enough to consider $I = [a_{i-1}, a_i)$ for $i \in A$. One has,

$$\begin{aligned}
&\mu_t(T_{(\lambda,\pi)}(I))\\
&= \frac{1}{K}\left(\sum_{n\geq 1} e^{\gamma(x_{-n}\dots x_{-1})}\delta_{T^{-n}_{(\lambda,\pi)}t} + \delta_t + \sum_{n\geq 1} e^{-\gamma(x_0\dots x_{n-1})}\delta_{T^{n}_{(\lambda,\pi)}t}\right)(T_{(\lambda,\pi)}(I))\\
&= \frac{1}{K}\left(\sum_{n\geq 1} e^{\gamma(x_{-n}\dots x_{-1})}\delta_{T^{-n-1}_{(\lambda,\pi)}t} + \delta_{T^{-1}_{(\lambda,\pi)}t} + \sum_{n\geq 1} e^{-\gamma(x_0\dots x_{n-1})}\delta_{T^{n-1}_{(\lambda,\pi)}t}\right)(I)\\
&= \frac{1}{K}\left(\sum_{n\geq 1} e^{-\gamma(x_{-n})}e^{\gamma(x_{-n}\dots x_{-1})}\delta_{T^{-n}_{(\lambda,\pi)}t} + e^{-\gamma(x_0)}\delta_t + \sum_{n\geq 1} e^{-\gamma(x_n)}e^{-\gamma(x_0\dots x_{n-1})}\delta_{T^{n}_{(\lambda,\pi)}t}\right)(I)\\
&= e^{-\gamma_i}\mu_t(I)
\end{aligned}$$

where in the last equality we use the fact that $T^n_{(\lambda,\pi)}(t) \in I$ if and only if $\gamma(x_n) = \gamma_i$. □

Define $g : [0,1) \to [0,1)$ by $g(s) = \mu_t([0,s])$. This function is nondecreasing, right continuous and has left limits. Let $i \in A$. Denote $a'_i = T(a_i)$ and define $b_i = \lim_{a\to a_i^-} g(a)$ and $b'_i = \lim_{a'\to(a'_i)^-} g(a')$. Then at interval $[b_{i-1}, b_i)$ define linearly

the AIET $f$ with image $[b'_{i-1}, b'_i)$. The slope vector of $f$ is $w = (e^{-\gamma_1}, \ldots, e^{-\gamma_r})$. Indeed,

$$\frac{b'_i - b'_{i-1}}{b_i - b_{i-1}} = \frac{\mu_t([a'_{i-1}, a'_i))}{\mu_t([a_{i-1}, a_i))} = e^{-\gamma_i}$$

where the last equality follows from Lemma 19.
Let $h : [0, 1) \to [0, 1)$ be the map defined by: $h(v) = u$ if $g(u) = v$ and $h(v) = u$ if $\lim_{w \to u^-} g(w) \leq v \leq g(u)$. Clearly $h$ is surjective, continuous and non decreasing. Since $\mu_t$ has atoms, then $h$ is not injective
The following lemma allows to conclude Theorem 1.

**Lemma 20.** *The map $h$ defines a semi-conjugacy between the AIET $f$ and $T_{(\lambda,\pi)}$. Moreover, $f$ has wandering intervals.*

*Proof.* The semi-conjugacy follows from construction. The interval

$$I = (\lim_{s \to t^-} g(s), g(t)]$$

is a wandering interval for $h$. □

## 5. Pseudo-Anosov diffeomorphisms and eigenvalues of matrices obtained by Rauzy induction

In this section, we discuss the hypothesis of Theorem 1 in a geometric language. Our hypothesis is that the Perron-Frobenius eigenvalue $\theta_1$ of the matrix $R$ has a real conjugate $\theta_2 > 1$.
We recall that every interval exchange transformation $T_{(\lambda,\pi)}$ is realized as the first return map of a flow on a translation surface $\mathcal{S}$ which genus $g(\pi)$ only depends on the permutation $\pi$ (and not on $\lambda$). This translation surface is not unique. If $T_{(\lambda,\pi)}$ is a periodic point of the Rauzy induction, one can choose $\mathcal{S}$ fixed by a pseudo-Anosov diffeomorphism $\phi$ (see [Th] for an enlightening discussion on pseudo-Anosov diffeomorphisms). The eigenvalue $\theta_1$ is the dominant eigenvalue of the action of $\phi$ on the absolute homology of $\mathcal{S}$. Therefore $\theta_1$ is an algebraic number of degree at most $2g(\pi)$ over $\mathbb{Q}$.
Heuristically, after the work of Avila and Viana [AV], it is reasonable to believe that a "generic" pseudo-Anosov satisfies our hypothesis. Nevertheless, it seems extremely difficult to understand the eigenvalues of *all* pseudo-Anosov diffeomorphisms. In this section, we want to explain that our hypothesis are often satisfied. They are not always satisfied: for instance, the conjugates of the Arnoux-Yoccoz pseudo-Anosov are not real. Situations much worse do exist.

### 5.1. Existence of a conjugate $\theta_2$ with $|\theta_2| \geq 1$.

A pseudo-Anosov diffeomorphism preserves the symplectic form induced by the intersection form. Thus if $z$ is an eigenvalue of the automorphism $\phi_*$ of $H_1(\mathcal{S}, \mathbb{Z})$, its inverse $z^{-1}$ is also an eigenvalue of $\phi_*$. Consequently, $\frac{1}{\theta_1}$ is an eigenvalue of $\phi_*$. If it is the only Galois conjugate of $\theta_1$, it means that $\theta_1$ is an algebraic number of degree 2. It is classical (see [KS] for instance) that the surface $\mathcal{S}$ is then a covering of a torus (a square tiled surface up to normalization). Therefore hypothesis (1) is satisfied if and only if the surface $\mathcal{S}$ is not a square tiled surface. Thus, this hypothesis is very natural and simple to check.

5.2. **Real conjugates.** The second hypothesis is more subtle to analyze.
A pseudo-Anosov diffeomorphism is obtained by Thurston's construction if it is the product of two affine Dehn twists $T_h$ and $T_v$ along two multi-curves filling a surface (see [Th]).
After normalization, the derivatives of the Dehn twists in the natural parameters of the translation surface are

$$T_h = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},\ T_v = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

where $a$ and $b$ are positive real numbers and $ab$ is an algebraic number.
An element $f$ of the group generated by $T_h$ and $T_v$ is a pseudo-Anosov diffeomorphism if the absolute value of the trace $t(f)$ of the corresponding matrix is larger than 2. For every pseudo-Anosov diffeomorphism obtained by Thurston's construction, the conjugates of $t(f)$ are real numbers (see [HL]). The dominant root of the action of $f$ on the homology is the real number $\theta_1 > 1$ with $\theta_1 + \theta_1^{-1} = t(f)$. The number $\theta_1$ (or one of its power) is the Perron-Frobenius eigenvalue of the matrix obtained by Rauzy induction considered in the present paper (see [Ve]). Let $\theta'$ be a conjugate of $\theta_1$ and $t'(f) = \theta' + \theta'^{-1}$ a (*real*) conjugate of $t(f)$. $\theta'$ is a real number with $\theta' > 1$ if $|t'(f)| > 2$. It is a complex number of modulus one if $|t'(f)| < 2$. This directly comes from the fact that $\theta' + \theta'^{-1} = t'(f)$.
For instance, the diffeomorphisms $f_{n,m} = T_h^n T_v^m$ are pseudo-Anosov diffeomorphisms if $n, m$ are positive integers. In fact the absolute value of the trace of $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}^m$ is larger than 2 because $nmab > 0$.
Thus $\theta' > 1$ if $|t'(f)| = |2 + nm(ab)'| > 2$ (where $(ab)'$ is a real number). This is satisfied for all couples $(n, m)$ except for finite number of exceptions. Using more sophisticated argument, $|t'(f)| > 2$ if $n$ and $m$ are positive integers.

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Institut de Mathématiques de Luminy, 163 avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France.
*E-mail address*: `bressaud@iml.univ-mrs.fr`

Laboratoire Analyse, Topologie et Probabilités, Case cour A, Faculté des Sciences de Saint-Jerôme, Avenue Escadrille Normandie-Niemen, 13397 Marseille Cedex 20, France.
*E-mail address*: `hubert@cmi.univ-mrs.fr`

Centro de Modelamiento Matemático and Departamento de Ingeniería Matemática, Universidad de Chile, Av. Blanco Encalada 2120, Santiago, Chile.
*E-mail address*: `amaass@dim.uchile.cl`